

SPACE-TIMES ADMITTING A THREE-DIMENSIONAL CONFORMAL GROUP

J. Carot*

A.A. Coley[†]

A.M. Sintes*

*Departament de Física, Universitat de les Illes Balears, E-07071 Palma de Mallorca. SPAIN.
†Department of Mathematics, Statistics and Computing Science, Dalhousie University, Halifax,
Nova Scotia. CANADA B3H 3J5.

Abstract

Perfect fluid space-times admitting a three-dimensional Lie group of conformal motions containing a two-dimensional Abelian Lie subgroup of isometries are studied. Demanding that the conformal Killing vector be proper (i.e., not homothetic nor Killing), all such space-times are classified according to the structure of their corresponding three-dimensional conformal Lie group and the nature of their corresponding orbits (that are assumed to be non-null). Each metric is then explicitly displayed in coordinates adapted to the symmetry vectors. Attention is then restricted to the diagonal case, and exact perfect fluid solutions are obtained in both the cases in which the fluid four-velocity is tangential or orthogonal to the conformal orbits, as well as in the more general “tilting” case.

PACS numbers: 04.20Jb, 02.40+m, 98.80Dr

1 Introduction

We shall study perfect fluid space-times admitting a three-dimensional Lie group of conformal motions C_3 , which contains a two-dimensional Abelian Lie subgroup of isometries G_2 . The precise assumptions are:

1. The G_2 acts orthogonally transitively on two-dimensional spacelike orbits diffeomorphic to \mathbb{R}^2 .
2. One of the Killing vectors (KV) in G_2 is hypersurface orthogonal, hence the metric can be written in diagonal form [1].
3. The conformal Killing vector (CKV) is proper (non-homothetic).
4. The orbits of the C_3 are non-null.

Space-times of this type have been studied recently by Kramer and Carot [2] and Castejón-Amenedo and Coley [3]. Here we shall systematically classify geometrically all possible space-times according to their (inequivalent) group structures and the nature of their corresponding orbits, and thereafter attempt to obtain the perfect fluid models in each class. In this classification scheme the (stationary and axisymmetric) models studied by Kramer [4] and Kramer and Carot [2] admit a C_3 acting on three-dimensional timelike hypersurfaces. The models studied by Castejón-Amenedo and Coley [3] admit a C_3 acting on two-dimensional timelike hypersurfaces containing an Abelian C_2 . Mars and Senovilla are looking at models in which C_3 is Abelian and trying to extend their study to other non-diagonal cases [5] and Coley and Czapor [6] are studying space-times in which the CKV is inheriting [7].

This work generalizes previous work in several ways. First, solutions admitting a CKV in space-times with additional symmetry have been studied; for example,

spatially homogeneous [7], spherically symmetric (see, for instance [8] and references cited therein) and plane symmetric [9] models have been investigated. In a sense the present work is the natural generalization of this research in that CKV are studied in space-times with the next highest degree of symmetry. Indeed, it is for this reason that G_2 space-times have begun to attract much attention. Cosmological models with an Abelian G_2 acting on spacelike hypersurfaces have been studied by Hewitt et al.[1], Hewitt and Wainwright [10], Hewitt et al.[11], Ruiz and Senovilla [12] and Van den Berg and Skea [13]. Models with an Abelian G_2 acting on timelike hypersurfaces, including the astrophysically relevant stationary and axisymmetric models, which have been studied for many years [14], have also attracted renewed attention [15, 16, 17, 18]. Second, the cases in which the CKV degenerates to either a KV or a homothetic vector field (HVF) have been studied previously. Space-times admitting a G_3 have received much attention [14]. Space-times admitting a HVF in addition to two KV have been studied by Hewitt and Wainwright [10] and Carot et al. [19]. Third, this work generalizes that of Coley and Czapor [6] in which the CKV is inheriting, and complements that of Senovilla [17] in which the Abelian G_2 acts on two-dimensional timelike hypersurfaces (these models are, in fact, stationary and axisymmetric perfect fluid solutions admitting a proper CKV). Finally, we shall assume that the perfect fluid matter satisfies $\mu > 0$ (all vacuum space-times admitting a proper CKV are known, [7, 20]).

For a very precise and accurate study of the general properties of CKVs, their Lie algebra and fixed point structure, we refer the reader to [21].

The outline of this paper is as follows. In section 2 we shall describe the models under consideration in detail, defining all relevant terms. We shall then classify the

space-times into different inequivalent classes, in each of the two cases in which the CKV is everywhere spacelike or not (i.e., the CKV can be timelike, see Eqns. (9) and (10)), according to the structure of their corresponding three-dimensional conformal Lie group C_3 (each containing an Abelian G_2). We shall display the corresponding metrics (and symmetry vectors) in coordinates adapted to the CKV. These results are completely independent of the Einstein field equations and the assumed energy-momentum tensor. We note parenthetically that not all the classes may be possible (again, irrespective of the assumed matter content) in some situations of physical interest; for example, Mars and Senovilla [18] have shown that in axially symmetric space-times admitting a maximal three-dimensional (or two-dimensional) conformal Lie algebra (such as in a cylindrically symmetric space-time with one CKV), the axial KV must commute with the other two (in fact, they show that this is true regardless of whether the additional KV is spacelike, such as, for example, in the case of a stationary and axially symmetric space-time with one CKV).

In section 3 we study those cases where one KV is hypersurface orthogonal, hence the metric can be written in diagonal form. In sections 4 and 5 we then utilize Einstein's field equations in an attempt to find perfect fluid solutions representative of the classes in each of the two cases corresponding to the form of the CKV. We shall assume that the matter satisfies the weak and dominant energy conditions ($\mu > 0$, $-\mu \leq p \leq \mu$). In the final section all of the perfect fluid solutions obtained are summarized and briefly discussed.

2 Space-times admitting conformal Killing vectors

As we have already pointed out, we shall concern ourselves with space-times (M, g) that admit a three-parameter conformal group C_3 containing an Abelian two-parameter

subgroup of isometries G_2 , whose orbits S_2 are spacelike, diffeomorphic to \mathbb{R}^2 and admit orthogonal two-surfaces; furthermore, we shall assume that the C_3 acts transitively on non-null orbits V_3 . As is customary, we shall denote the KV spanning \mathcal{G}_2 (the Lie algebra associated with G_2) by ξ and η , and the proper CKV in \mathcal{C}_3 (the Lie algebra of the conformal group C_3) by X . Since, by hypothesis, ξ and η commute, we can (locally) adapt two coordinates, say y and z , so that

$$\xi = \frac{\partial}{\partial y}, \quad \eta = \frac{\partial}{\partial z}. \quad (1)$$

Taking now two more coordinates t and r , it follows from the above assumptions that the line element associated with the metric g can be written as [22]

$$ds^2 = e^{2F} \{-dt^2 + dr^2 + Q[h^{-2}(dy + wdz)^2 + h^2 dz^2]\}, \quad (2)$$

where F , Q , h and w are all functions of t and r alone.

The (proper) CKV X will satisfy

$$(\mathcal{L}_X g)_{ab} = 2\Psi g_{ab}, \quad (3)$$

where \mathcal{L} stands for the Lie derivative operator and $\Psi = \Psi(x^c)$ is the so-called conformal factor (the particular cases $\Psi = 0$ and $\Psi = const \neq 0$ correspond, respectively, to X being a KV and a proper HVF). Assuming that no further CKV exist on M (i.e., the C_3 is maximal) one has the following two families of Lie algebra structures for \mathcal{C}_3 :

$$(A) \quad [\xi, \eta] = 0, \quad [\xi, X] = \alpha_1 \xi + \alpha_2 \eta, \quad [\eta, X] = \beta_1 \xi + \beta_2 \eta, \quad (4)$$

$$(B) \quad [\xi, \eta] = 0, \quad [\xi, X] = a_1 \xi + a_2 \eta + a_3 X, \quad [\eta, X] = b_1 \xi + b_2 \eta + b_3 X, \quad (5)$$

where the α_i , β_i , a_i and b_i are constants.

The family $\mathcal{C}_3(A)$ can, in turn, be classified into the following seven Bianchi types (see for instance [23]):

- $$\begin{aligned}
(I) \quad & [\xi, \eta] = [\xi, X] = [\eta, X] = 0 , \\
(II) \quad & [\xi, \eta] = [\xi, X] = 0 \quad [\eta, X] = \xi , \\
(III) \quad & [\xi, \eta] = 0 \quad [\xi, X] = \xi \quad [\eta, X] = 0 , \\
(IV) \quad & [\xi, \eta] = 0 \quad [\xi, X] = \xi \quad [\eta, X] = \xi + \eta , \\
(V) \quad & [\xi, \eta] = 0 \quad [\xi, X] = \xi \quad [\eta, X] = \eta , \\
(VI) \quad & [\xi, \eta] = 0 \quad [\xi, X] = \xi \quad [\eta, X] = q\eta \quad (q \neq 0, 1) , \\
(VII) \quad & [\xi, \eta] = 0 \quad [\xi, X] = \eta \quad [\eta, X] = -\xi + q\eta \quad (q^2 < 4) ,
\end{aligned} \tag{6}$$

whereas family $\mathcal{C}_3(B)$ can be brought to the following form by means of appropriate re-definitions of X and the KV's ξ and η :

$$[\xi, \eta] = 0 \quad [\xi, X] = X \quad [\eta, X] = 0 . \tag{7}$$

Notice that in the case of X being a HVF the above algebraic structure (7) is forbidden, since the Lie bracket of a HVF and a KV must necessarily be a KV (for further information on this case, see [19]).

Assuming now for the CKV X an expression of the form

$$X = X^a(x^c)\partial_a , \tag{8}$$

in the coordinate chart $\{t, r, y, z\}$, and specializing the equation (3) to the metric given in (2) and the CKV given in (8), it is easy to see that one can always, by means of a coordinate transformation in the t, r plane, bring X to one of the following forms:

- $$\begin{aligned}
(a) \quad & X = \partial_t + X^y(y, z)\partial_y + X^z(y, z)\partial_z , \\
(b) \quad & X = \partial_r + X^y(y, z)\partial_y + X^z(y, z)\partial_z , \\
(c) \quad & X = \partial_t + \partial_r + X^y(y, z)\partial_y + X^z(y, z)\partial_z ,
\end{aligned} \tag{9}$$

if the conformal algebra \mathcal{C}_3 belongs to the family (A), whence $X^y(y, z)$ and $X^z(y, z)$ are linear functions of their arguments to be determined from the commutation relations of X with ξ and η (see (6-(I)) to (6-(VII)); and to the forms:

$$\begin{aligned} (a) \quad X &= e^y(\partial_t + X^y(t)\partial_y + X^z(t)\partial_z) , \\ (b) \quad X &= e^y(\partial_r + X^y(r)\partial_y + X^z(r)\partial_z) , \\ (c) \quad X &= e^y(\partial_t + \partial_r + X^y(t, r)\partial_y + X^z(t, r)\partial_z) , \end{aligned} \quad (10)$$

if \mathcal{C}_3 is that given by (7) (family (B)).

The forms (9-a) and (10-a) are easily seen to correspond to the case of three-dimensional timelike conformal orbits T_3 , whereas (9-b) and (10-b) correspond to three-dimensional spacelike conformal orbits S_3 . The remaining possibilities, (9-c) and (10-c), imply null conformal orbits N_3 and will not be considered in the present paper.

Assuming now the form (9-a) for the CKV X (i.e., family (A), timelike conformal orbits), one then has for each possible case (I) to (VII) the following forms for X and the metric functions:

$$\begin{aligned} (I) \quad Q &= \hat{Q}(r) , \quad h^2 = \hat{h}^2(r) , \quad w = \hat{w}(r) , \\ X &= \partial_t . \end{aligned} \quad (11)$$

$$\begin{aligned} (II) \quad Q &= \hat{Q}(r) , \quad h^2 = \hat{h}^2(r) , \quad w = \hat{w}(r) - t , \\ X &= \partial_t + z\partial_y . \end{aligned} \quad (12)$$

$$\begin{aligned} (III) \quad Q &= e^{-t}\hat{Q}(r) , \quad h^2 = e^t\hat{h}^2(r) , \quad w = e^t\hat{w}(r) , \\ X &= \partial_t + y\partial_y . \end{aligned} \quad (13)$$

$$\begin{aligned} (IV) \quad Q &= e^{-2t}\hat{Q}(r) , \quad h^2 = \hat{h}^2(r) , \quad w = \hat{w}(r) - t , \\ X &= \partial_t + (y+z)\partial_y + z\partial_z . \end{aligned} \quad (14)$$

$$(V) \quad Q = e^{-2t}\hat{Q}(r) , \quad h^2 = \hat{h}^2(r) , \quad w = \hat{w}(r) ,$$

$$X = \partial_t + y\partial_y + z\partial_z . \quad (15)$$

$$(VI) \quad Q = e^{-(1+q)t}\hat{Q}(r) , \quad h^2 = e^{(1-q)t}\hat{h}^2(r) , \quad w = e^{(1-q)t}\hat{w}(r) ,$$

$$X = \partial_t + y\partial_y + qz\partial_z \quad (q \neq 0, 1) . \quad (16)$$

$$(VII) \quad Q = e^{-qt}\frac{\sqrt{4-q^2}}{2}a(r) ,$$

$$h^2 = \frac{\frac{\sqrt{4-q^2}}{2}a(r)}{\sqrt{a(r)^2 + c(r)^2 + g(r)^2 + c(r)\cos(\sqrt{4-q^2}t) + g(r)\sin(\sqrt{4-q^2}t)}} ,$$

$$w = \frac{q}{2} + \frac{\frac{\sqrt{4-q^2}}{2}[c(r)\sin(\sqrt{4-q^2}t) - g(r)\cos(\sqrt{4-q^2}t)]}{\sqrt{a(r)^2 + c(r)^2 + g(r)^2 + c(r)\cos(\sqrt{4-q^2}t) + g(r)\sin(\sqrt{4-q^2}t)}} ,$$

$$X = \partial_t - z\partial_y + (y + qz)\partial_z \quad (q^2 < 4) . \quad (17)$$

In all of these cases $F = F(t, r)$ and the conformal factor, Ψ , is given by

$$\Psi = F_{,t} \quad (18)$$

The form (9-b) for the CKV X (i.e., family A, spacelike conformal orbits) would yield similar results with the role of the coordinates t and r reversed.

Similarly, if we take the form (10-a) for X (Family B, timelike conformal orbits) we get, assuming the canonical form (7) for \mathcal{C}_3 , the following possibilities for the metric and the CKV:

$$(1) \quad ds^2 = e^{2F}\{-dt^2 + dr^2 + [\Phi^2(r)(c\cosh t + d)^2 + \sinh^2 t]dy^2 +$$

$$+ 2\Phi^2(r)(c\cosh t + d)dydz + \Phi^2(r)dz^2\} ,$$

$$X = e^y \left(\partial_t - \coth t\partial_y + \frac{c + d\cosh t}{\sinh t}\partial_z \right) , \quad (19)$$

$$(2) \quad ds^2 = e^{2F}\{-dt^2 + dr^2 + [\Phi^2(r)(ct^2 + d)^2 + t^2]dy^2 +$$

$$+ 2\Phi^2(r)(ct^2 + d)dydz + \Phi^2(r)dz^2\} ,$$

$$X = e^y \left(\partial_t - \frac{1}{t}\partial_y - \frac{ct^2 - d}{|t|}\partial_z \right) , \quad (20)$$

$$(3) \quad ds^2 = e^{2F}\{-dt^2 + dr^2 + [\Phi^2(r)(d - c\cos t)^2 + \sin^2 t]dy^2 +$$

$$+2\Phi^2(r)(d - c \cos t)dydz + \Phi^2(r)dz^2\} , \\ X = e^y \left(\partial_t - \cot t \partial_y + \frac{d \cos t - c}{\sin t} \partial_z \right) , \quad (21)$$

where c and d are constants, $F = F(t, r)$ is an arbitrary function of t and r , and the conformal factor Ψ is given by

$$\Psi = e^y F_{,t} . \quad (22)$$

The case of spacelike conformal orbits for the family B (i.e., X of the form (10-b)) leads to just one possibility, namely

$$ds^2 = e^{2F} \{-dt^2 + dr^2 + [\Phi^2(t)(c \sinh r + d)^2 + \cosh^2 r]dy^2 + \\ + 2\Phi^2(t)(c \sinh r + d)dydz + \Phi^2(t)dz^2\} , \\ X = e^y \left(\partial_r - \tanh r \partial_y + \frac{c - d \sinh r}{\cosh r} \partial_z \right) , \quad (23)$$

where c and d are again constants, $F = F(t, r)$ is an arbitrary function of its variables and the conformal factor is in this case

$$\Psi = e^y F_{,r} . \quad (24)$$

3 Diagonal Case.

Next we will study those cases where two hypersurface orthogonal KVs exist in \mathcal{G}_2 ; this implies that the two KVs must be mutually orthogonal and therefore, by means of a linear change of coordinates in the Killing orbits S_2 , one can always set w in (2) to zero and the metric then becomes diagonal [22].

It is worth noticing that this is not possible for types *II* and *IV* in family A, as one can see by simply inspecting the form of w in these cases (see (12) and (14)). As for the case *VII* (also belonging to family A), $w = 0$ implies the existence of a third KV tangent to the Killing orbits S_2 , in which case they are of constant curvature and

the conformal algebra becomes four dimensional; therefore, we shall not consider this case further. For those metrics in family B we have:

$$\begin{aligned} ds^2 &= e^{2F} \{-dt^2 + dr^2 + \sinh^2 t dy^2 + \Phi^2(r) dz^2\}, \\ X &= e^y (\partial_t - \coth t \partial_y), \end{aligned} \quad (25)$$

$$\begin{aligned} ds^2 &= e^{2F} \{-dt^2 + dr^2 + t^2 dy^2 + \Phi^2(r) dz^2\}, \\ X &= e^y \left(\partial_t - \frac{1}{t} \partial_y \right), \\ ds^2 &= e^{2F} \{-dt^2 + dr^2 + \sin^2 t dy^2 + \Phi^2(r) dz^2\}, \end{aligned} \quad (26)$$

$$X = e^y (\partial_t - \cot t \partial_y), \quad (27)$$

if the conformal orbits are timelike, and, for spacelike conformal orbits:

$$\begin{aligned} ds^2 &= e^{2F} \{-dt^2 + dr^2 + \cosh^2 r dy^2 + \Phi^2(t) dz^2\}, \\ X &= e^y (\partial_r - \tanh r \partial_y). \end{aligned} \quad (28)$$

There is still another possibility for metrics belonging to this family; namely,

$$\begin{aligned} ds^2 &= e^{2F} \{-dt^2 + dr^2 + \cos^2 t dy^2 + \sin^2 t dz^2\}, \\ X &= e^{y+z} (\partial_t + \tan t \partial_y - \cot t \partial_z), \end{aligned} \quad (29)$$

but this metric admits a further proper CKV, ∂_r , therefore the maximal conformal group is four-dimensional and again we shall not consider this case any further.

4 Perfect fluid space-times: Fluid flow tangent or orthogonal to the conformal orbits.

In this section we will study diagonal perfect fluid solutions admitting a maximal C_3 of conformal motions. The energy-momentum tensor for a perfect fluid is given by

$$T_{ab} = (\mu + p) u_a u_b + p g_{ab}, \quad (30)$$

where μ and p are, respectively, the energy density and the pressure as measured by observers comoving with the fluid, and u^a ($u^a u_a = -1$) is the four-velocity of the fluid. Since the metric admits two KVs, ξ and η , it follows that the Lie derivative of p , μ and u_a with respect to each must vanish identically; in our coordinate chart this is equivalent to:

$$\mu = \mu(t, r) \quad p = p(t, r) \quad u_a = u_a(t, r) . \quad (31)$$

Since the isometry orbits admit orthogonal two-surfaces one has that, in these coordinates, both the metric and the Ricci tensor have block-diagonal forms; thus, the Einstein field equations specialized to a perfect fluid, as given by (30), reduce simply to

$$u_y = u_z = 0 , \quad (32)$$

$$\frac{G_{yy}}{g_{yy}} - \frac{G_{zz}}{g_{zz}} = 0 , \quad (33)$$

$$G_{tr}^2 - \left(G_{tt} - \frac{G_{yy}}{g_{yy}} g_{tt} \right) \left(G_{rr} - \frac{G_{yy}}{g_{yy}} g_{rr} \right) = 0 \quad (34)$$

(where we have already taken into account that the metric is diagonal, otherwise we would have had an extra equation, namely: $G_{yy}/g_{yy} - G_{yz}/g_{yz} = 0$).

We shall focus on those solutions such that the perfect fluid four-velocity is either tangent or orthogonal to the conformal orbits; the general case (or “tilted” case) will be the subject of the next section.

4.1 Fluid flow tangent to the conformal orbits.

This case can only arise when the conformal orbits are timelike, and then one has:

$$u_t = -e^F \quad u_r = 0 \quad (35)$$

(i.e., the coordinates are comoving). The metric must be one of those given by (11), (13), (15) and (16) (with $w = 0$) if belongs to family A (Bianchi types I , III , V and

VI, respectively), or one of those given by (25), (26) and (27) if it belongs to family B.

We next summarize the results obtained under these hypotheses for metrics of both families A and B.

Family A:

Type *I*: In this case, the fluid velocity is parallel to the CKV X . This case has already been studied in full generality (see [24]) showing that the only such space-times are locally FRW models (assuming $\mu + p \neq 0$ and $p = p(\mu)$) and therefore the C_3 is not maximal. Also, without making any assumptions on the equation of state, it can be easily seen that either the C_3 is not maximal or X is not a proper CKV.

Type *III*: The only solution of this type which admits no further CKVs (including KVs and HVFs) is

$$ds^2 = e^{2f} \left\{ -dt^2 + dr^2 + \frac{e^{-2t}}{(\cosh r)^{2(A+1)}} dy^2 + \frac{\sinh^2 r}{(\cosh r)^{2(A+1)}} dz^2 \right\} , \quad (36)$$

where

$$f = -\frac{A}{2}U - \ln(\cosh kU) + A \ln(\cosh r) + \alpha , \quad (37)$$

A and α are constants, $U = U(t, r)$ is given by

$$U = t + \ln(\cosh r) , \quad (38)$$

the constant k takes the value,

$$k = \frac{\sqrt{(A+1)^2 + 1}}{2} , \quad (39)$$

and μ and p are given by,

$$\mu e^{2f} = \frac{1}{\cosh^2 r} \{1 + A + 3(M - 1)^2\}, \quad (40)$$

$$p e^{2f} = \frac{1}{\cosh^2 r} \{-1 + 2A - 3M^2 + (4 - 2A)M\}, \quad (41)$$

where $M = M(U)$ is defined by

$$M \equiv -\frac{A}{2} - k \tanh(kU). \quad (42)$$

The imposition of the energy conditions ($\mu > 0$, $-\mu \leq p \leq \mu$) results in restrictions on the values that the parameter A can take, namely $A \in (-1, 5)$.

Type V: In this case X cannot be a proper CKV. However, there are solutions belonging to this type for which X is a HVF (see for instance [19]).

Type VI: No perfect fluid solutions of this type exist which admit a proper CKV, as can easily be seen from the field equations.

Family B:

All the solutions in this family which satisfy the conditions set up at the beginning of this subsection are either such that X becomes a KV (metrics of the form (27)), or they admit two further KV which together with ξ and η generate a four parameter group of isometries G_4 acting on three-dimensional spacelike orbits, the space-time thus becoming spatially homogeneous (metrics of the form (25) and (26)). Furthermore, the subgroup G_3 that G_4 necessarily contains acts on two-dimensional spacelike orbits, coordinated by r and z , which are then of positive constant curvature; hence the models are spherically symmetric Kantowski-Sachs space-times.

4.2 Fluid flow orthogonal to the conformal orbits.

The conformal orbits must be spacelike in this case, and the coordinates are comoving, i.e.

$$u_t = -e^F \quad u_r = 0 . \quad (43)$$

Family A:

Type *I*: The only solutions belonging to this type which admit no further symmetry are

$$ds^2 = \frac{f_o^2}{\cosh^2 r} \left\{ -dt^2 + dr^2 + \frac{(\cosh t)^{\sigma+1}}{(\sinh t)^{\sigma-1}} dy^2 + \frac{(\sinh t)^{\sigma+1}}{(\cosh t)^{\sigma-1}} dz^2 \right\}, \quad (44)$$

where f_o is an arbitrary constant and the parameter σ must be such that $|\sigma| < 1$; the energy density and pressure are then given by

$$\mu = \frac{1}{f_o^2} \left\{ (1 - \sigma^2) \frac{\cosh^2 r}{\sinh^2 2t} + 3 \right\}, \quad (45)$$

$$p = \frac{1}{f_o^2} \left\{ (1 - \sigma^2) \frac{\cosh^2 r}{\sinh^2 2t} - 3 \right\}. \quad (46)$$

The equation of state is simply $p = \mu - 6/f_o^2$ as can be easily seen from (45) and (46). Notice that this solution is only valid in the region $t > 0$ and that it is separable in the variables t and r ; and hence it must be contained in the solutions given by Ruiz and Senovilla [12].

Type *III*: No perfect fluid solutions admitting a proper CKV exist, since $\mu + p = 0$.

Type *V*: There are no solutions of this type for perfect fluids with $\mu > 0$ and the conditions set up above, as can easily be seen from the field equations.

Type *VI*: In this case, the line element, density and pressure take the following forms:

$$ds^2 = e^{2f(t,r)} \left\{ -dt^2 + dr^2 + Q(t) \left[\frac{e^{-2r}}{h^2(t)} dy^2 + e^{-2qr} h^2(t) dz^2 \right] \right\}, \quad (47)$$

$$\mu e^{2f} = (\dot{\beta}^2 - 1) \left\{ \frac{3m^2}{(1 + e^{-mu})^2} + qn^2 \frac{1}{\dot{\beta}^2} \right\}, \quad (48)$$

$$pe^{2f} = (\dot{\beta}^2 - 1) \left\{ qn^2 \frac{1}{\dot{\beta}^2} - \frac{3m^2}{(1 + e^{-mu})^2} + 4q \frac{1+q^2}{(1+q)^2} \frac{1}{1+e^{-mu}} \right\}, \quad (49)$$

where $q \neq 0, 1$ and

$$m \equiv \frac{1+q^2}{1+q} \quad n \equiv \frac{1-q}{1+q}, \quad (50)$$

and

$$f = H(u) + \lambda(t) \quad u \equiv r + \beta(t), \quad (51)$$

where $H(u)$ is given by

$$H = -\ln(1 + e^{-mu}), \quad (52)$$

and there are two possibilities for $\beta(t)$, namely:

$$(i) \quad \beta = -\frac{1}{k} \ln |\sinh kt|, \quad (53)$$

with $k \equiv -\frac{(q-1)^2}{q+1}$, whence

$$Q = \cosh kt \quad h^2 = (\cosh kt)^{1/n}, \quad (54)$$

$$\lambda = -\frac{1+q^2}{(1-q)^2} \ln |\sinh kt| + \lambda_o, \quad \lambda_o = const. \quad (55)$$

$$(ii) \quad \beta = -\frac{1}{k} \ln(\cosh kt), \quad (56)$$

whence

$$Q = \sinh kt \quad h^2 = (\sinh kt)^{1/n}, \quad (57)$$

$$\lambda = -\frac{1+q^2}{(1-q)^2} \ln(\cosh kt) + \lambda_o, \quad \lambda_o = const. \quad (58)$$

Again, these solutions are only valid for $t > 0$. It is also easy to see that the choice of a negative value for the parameter q would imply that $(\mu + p)$ and $(\mu - p)$ have opposite signs, thus violating one of the energy conditions (furthermore, μ could only be non-negative in a certain region of the space-time). On the other hand, assuming $q > 0$ immediately leads to the choice (i) for $\beta(t)$ in order to have $\mu > 0$, whence $\mu + p$ is always non-negative but there will always be some region of the space-time in which $\mu - p < 0$; consequently these solutions are only valid in a certain region of the space-time (in particular: (t, r) satisfying $\sinh |k|t < [(q^2 - 2q/3 + 1)/2q]^{k/m} e^{kr}$).

Family B:

It is easy to see from the field equations that there are no perfect fluid solutions in this family.

5 Perfect fluid space-times: Tilted case.

The field equations in this case are (33) and (34), but we no longer have the additional condition $u_r = 0$ as in the previous cases. However, it should be noted that for a perfect fluid solution admitting an Abelian G_2 of isometries acting on spacelike orbits and such that they admit orthogonal surfaces, it is always possible to perform a change of coordinates so as to bring the coordinates into a comoving form with respect to u^a , while leaving the metric diagonal [22]. Such a coordinate change in the t, r plane (in our coordinates) would dramatically change the form of the proper CKV X ; for

example, we would no longer be able to integrate out the conformal equations (3) nor provide simple expressions for the metric functions. Roughly speaking, one must choose between the coordinate chart adapted to the CKV and the one adapted to the four-velocity of the fluid. It is interesting to notice that this is not the case for HVFs; since one may change there to comoving coordinates the HVF changing then in a “controlled” way, so that one can still integrate out the homothetic equations (see for instance [19]). This difference is mainly due to the fact that the four-velocity field of a perfect fluid and a HVF are always surface forming (or in a more physical language, the fluid “inherits” the symmetry, see [7]), whereas this is not necessarily so in the case of a CKV.

We shall deal separately with the cases of timelike and spacelike conformal orbits and, in each case, we shall distinguish between the two families of metrics A and B.

5.1 Timelike conformal orbits.

Family A:

We shall discuss here some general features of the solutions belonging to different types in this family.

The metric can be written as:

$$ds^2 = e^{2F(t,r)} \left\{ -dt^2 + dr^2 + e^{at} \frac{q(r)}{h^2(r)} dy^2 + e^{bt} q(r) h^2(r) dz^2 \right\}, \quad (59)$$

where

Type	a	b
<i>I</i>	0	0
<i>III</i>	-2	0
<i>V</i>	-2	-2
<i>VI</i>	-2	-2q

and $q \neq 0, 1$.

For these cases Eqn. (33) gives:

$$\frac{a^2 - b^2}{4} - 2 \left(\frac{h'}{h} \right)^2 + 2 \frac{h' q'}{hq} + 2 \frac{h''}{h} + 4 \frac{h'}{h} F_{,r} + (a - b) F_{,t} = 0 , \quad (60)$$

where a dash indicates differentiation with respect to r . At this point we must distinguish between two cases depending on whether $a = b$ or not.

Case (i): Assume $a = b$ (types I and V). Equation (60) reads now:

$$0 = \left(\frac{h'}{h} \right)' + \frac{h' q'}{hq} + 2 \frac{h'}{h} F_{,r} \quad (61)$$

since $h' \neq 0$ (otherwise the metric admits a further KV), this equation immediately gives

$$e^{2F} = f(t) \frac{h}{qh'} \quad (62)$$

Case (ii): If $a \neq b$ (types III and VI). Differentiating (60) with respect to t , we have

$$0 = 4 \frac{h'}{h} F_{,rt} + (a - b) F_{,tt} , \quad (63)$$

that integrates to give

$$F = H(t + \beta(r)) + \lambda(r) , \quad (64)$$

with

$$\beta' \frac{h'}{h} = \frac{b - a}{4} , \quad (65)$$

and the whole equation (60) reads

$$\frac{a^2 - b^2}{4} + 2 \left(\frac{h'}{h} \right)' + 2 \frac{h' q'}{hq} + 4 \frac{h'}{h} \lambda' = 0 . \quad (66)$$

Let us deal next with the first case. For types I and V we can use an alternative form of the metric that satisfies (33) identically, in order to simplify the resulting

expressions:

$$ds^2 = \frac{e^{f(t)-q(r)}}{h'(r)} \left\{ -dt^2 + dr^2 \right\} + \frac{e^{f(t)+at}}{h'(r)} \left\{ e^{-h(r)} dy^2 + e^{h(r)} dz^2 \right\}, \quad (67)$$

the conformal factor now being:

$$\Psi = \frac{f_{,t}}{2}. \quad (68)$$

Equation (34) can be written as

$$0 = \Sigma_0 + a\Sigma_1 f_{,t} + \Sigma_2 f_{,t}^2 + \frac{a}{4} f_{,t}^3 + \Sigma_3 f_{,tt} - \frac{a}{2} f_{,t} f_{,tt}, \quad (69)$$

where Σ_i , $i = 0, \dots, 3$, denote functions depending only on r

$$\begin{aligned} \Sigma_0 &= \frac{a^2}{4}(q')^2 + \frac{a^2}{4} \frac{h''}{h'} q' + \frac{1}{4} h'' h' q' - \frac{1}{4} \left(\frac{h''}{h'} \right)^2 (q')^2 + \frac{1}{4} \left(\frac{h''}{h'} \right)^3 q' - \frac{a^2}{4} q'' \\ &\quad - \frac{1}{4} (h')^2 q'' - \frac{1}{4} \left(\frac{h''}{h'} \right)^2 q'' + \frac{1}{4} (q'')^2 - \frac{1}{2} \frac{h''' h''}{(h')^2} q' + \frac{1}{2} \frac{h'''}{h'} q'', \end{aligned} \quad (70)$$

$$\Sigma_1 = \frac{a^2}{4} + \frac{1}{4} (h')^2 + \frac{1}{2} (q')^2 + \frac{1}{2} \frac{h''}{h'} q' + \frac{1}{4} \left(\frac{h''}{h'} \right)^2 - \frac{1}{2} q'' - \frac{1}{2} \frac{h'''}{h'}, \quad (71)$$

$$\Sigma_2 = \frac{a^2}{2} + \frac{1}{4} (h')^2 + \frac{1}{4} (q')^2 + \frac{1}{4} \frac{h''}{h'} q' + \frac{1}{2} \left(\frac{h''}{h'} \right)^2 - \frac{1}{4} q'' - \frac{1}{2} \frac{h'''}{h'}, \quad (72)$$

$$\Sigma_3 = -\frac{a^2}{2} - \frac{1}{2} (h')^2 + \frac{1}{2} \frac{h''}{h'} q' - \frac{1}{2} \left(\frac{h''}{h'} \right)^2 + \frac{1}{2} q'' + \frac{h'''}{h'}, \quad (73)$$

Since $f_{,tt} \neq 0$ (otherwise X would become a HVF) the following possibilities arise from Eqn. (69) namely:

(I.a) $\Sigma_3 = 0$ then $\Sigma_2 = \Sigma_0 = 0$ (for all $f(t)$).

(I.b) $\Sigma_3 \neq 0$ then $\Sigma_2 = -c\Sigma_3$, $\Sigma_0 = \alpha\Sigma_3$,

$$0 = \alpha - c(f_{,t})^2 + f_{tt} \quad (74)$$

where α and c are constants.

(V.a) $\Sigma_i = \alpha_i$ (constants).

(V.b)

$$0 = e + df_{,t} + \frac{1}{2}(f_{,t})^2 - f_{,tt} \quad (75)$$

$$0 = 2\Sigma_2 + \Sigma_3 + 2d, \quad (76)$$

$$0 = \Sigma_1 + d\Sigma_3 + e, \quad (77)$$

$$0 = \Sigma_0 + e\Sigma_3, \quad (78)$$

where e, d are constants.

For all these cases one has:

$$G_{tr} = -\frac{a}{2}q' - \frac{1}{2}f_{,t}\left(q' + \frac{h''}{h'}\right), \quad (79)$$

$$\mu g_{rr} = \frac{3}{4}a^2 + \frac{3}{2}af_{,t} + \frac{3}{4}(f_{,t})^2 + \frac{1}{4}(h')^2 - \frac{3}{4}\left(\frac{h''}{h'}\right)^2 - \frac{1}{2}q'', \quad (80)$$

$$(\mu - p)g_{rr} = a^2 + 2af_{,t} + (f_{,t})^2 + f_{,tt} - 2\left(\frac{h''}{h'}\right)^2 + \frac{h'''}{h'}, \quad (81)$$

$$(\mu + p)g_{rr} = \frac{a^2}{2} + af_{,t} + \frac{1}{2}(f_{,t})^2 - f_{,tt} + \frac{1}{2}(h')^2 + \frac{1}{2}\left(\frac{h''}{h'}\right)^2 - q'' - \frac{h'''}{h'}, \quad (82)$$

$$(\mu + p)u_t^2 = \frac{a}{2}f_{,t} + \frac{1}{2}(f_{,t})^2 - f_{,tt} + \frac{1}{2}\frac{h''}{h'}q' - \frac{1}{2}q''. \quad (83)$$

Subcase I.a $a = 0$ and $\Sigma_0 = \Sigma_2 = \Sigma_3 = 0$. It can be trivially checked that

$$q' + \frac{h''}{h'} = 0, \quad (84)$$

and that $G_{tr} = 0$, so the solution has a comoving fluid flow and, as was pointed out before, it has more symmetry.

Subcase I.b $\Sigma_2 = -c\Sigma_3$ yields

$$0 = (1-2c)(h')^2 + (q')^2 + (1+2c)\frac{h''}{h'}q' + 2(1-c)\left(\frac{h''}{h'}\right)^2 - (1-2c)q'' - 2(1-2c)\frac{h'''}{h'}. \quad (85)$$

For the particular case $c = \frac{1}{2}$ the above equation reads

$$\left(q' + \frac{h''}{h'} \right)^2 = 0 , \quad (86)$$

and then $\Sigma_0 = \alpha\Sigma_3$ implies

$$\frac{1}{2} \frac{h'''}{h'} - \left(\frac{h''}{h'} \right)^2 = -\alpha , \quad (87)$$

and one can see from Eqn. (83) that $(\mu + p)u_t^2 = 0$, so, this solution cannot represent a perfect fluid. For $c \neq \frac{1}{2}$, we obtain, after some calculations involving (85)

$$\Sigma_0 = \frac{1}{4(1-2c)} \left[q'' - q' \frac{h''}{h'} \right] \left[q' + \frac{h''}{h'} \right]^2 , \quad (88)$$

$$\Sigma_3 = \frac{1}{2(1-2c)} \left[q' + \frac{h''}{h'} \right]^2 . \quad (89)$$

The remaining equation $\Sigma_0 = \alpha\Sigma_3$ can be written as

$$q'' - q' \frac{h''}{h'} = 2\alpha . \quad (90)$$

Defining now

$$\sigma' \equiv \frac{1}{h'} , \quad (91)$$

equation (90) can be integrated and (85) rewritten to give

$$q' = 2\alpha \frac{\sigma}{\sigma'} + \frac{k}{\sigma'} , \quad k = \text{const} , \quad (92)$$

$$0 = \sigma' \sigma''' + \frac{-1+3c}{1-2c} (\sigma'')^2 - \frac{2c}{1-2c} \sigma'' (2\alpha\sigma + k) - \alpha(\sigma')^2 + \frac{1}{2(1-2c)} (2\alpha\sigma + k)^2 + \frac{1}{2} , \quad (93)$$

and one then has

$$(\mu + p)u_t^2 = \frac{1-2c}{2} (f_{,t})^2 , \quad (94)$$

$$(\mu + p)g_{rr} = \frac{1-2c}{2} (f_{,t})^2 - \frac{1}{2(1-2c)} \left[2\alpha \frac{\sigma}{\sigma'} + \frac{k}{\sigma'} - \frac{\sigma''}{\sigma'} \right]^2 , \quad (95)$$

$$(\mu - p)g_{rr} = (1+c)(f_{,t})^2 - \alpha - \frac{\sigma'''}{\sigma'} . \quad (96)$$

In order to satisfy the energy conditions in all regions of the space-time, the right-hand sides of the above equations must all be positive; necessary conditions for this are $c < \frac{1}{2}$ and also that $(f_{,t})^2$ has a strictly lower bound. The latter condition singles out one of the possible solutions to (74) namely:

$$\alpha = c\beta^2 \quad f = -\frac{1}{c} \ln |\sinh(c\beta t + C_1)| . \quad (97)$$

where another additive constant of integration in the expression of f has been set equal to zero without loss of generality. Notice that C_1 can also be made zero by suitably redefining t , nevertheless we choose to maintain it in order to allow singularity-free solutions when $t = 0$.

One particular solution to (93) is

$$\sigma = \frac{\cosh \beta r}{\beta^2} \quad \text{and} \quad k = 1 - 2c . \quad (98)$$

Notice that in order to have the correct signature in the metric r is restricted to positive values only. The energy condition (96) is satisfied for all values of r but only if $c > -1$.

Type V: After some calculations, it can be seen that the only possibility for the line element given by (67) leading to an Einstein tensor of the perfect fluid type is:

$$ds^2 = e^{f(t)-2ar} \{-dt^2 + dr^2\} + e^{f(t)-2t} \{e^{-mr} dy^2 + e^{mr} dz^2\} \quad (99)$$

where a and m are constants. The only remaining field equation is then

$$0 = 4a^2 - (2 + 4a^2 + \frac{m^2}{2})f_{,t} + (2 + a^2 + \frac{m^2}{4})(f_{,t})^2 - \frac{1}{2}(f_{,t})^3 + f_{,tt}(f_{,t} - 2 - \frac{m^2}{2}), \quad (100)$$

that integrates to give

$$ke^t = (f_{,t} - 2)(f_{,t} - p^2(1 + s))^{(1-s)/2s} (f_{,t} - p^2(1 - s))^{-(1+s)/2s}, \quad (101)$$

where the constants p^2 and s are defined in terms of a and m by

$$p^2 = a^2 + \frac{m^2}{4} + 1 , \quad s^2 = 1 - \frac{4a^2}{p^4} . \quad (102)$$

The density, pressure and velocity field are given by

$$(\mu + p)u_t^2 = \frac{a^2(f_{,t} - 2)^2}{f_{,t} - 2(p^2 - a^2)} , \quad (103)$$

$$(\mu + p)g_{rr} = -(f_{,t} - 2(p^2 - a^2)) + \frac{a^2(f_{,t} - 2)^2}{f_{,t} - 2(p^2 - a^2)} , \quad (104)$$

$$(\mu - p)g_{rr} = \frac{(f_{,t} - 2)(\frac{3}{2}(f_{,t})^2 + (-3p^2 + 2a^2 - 2)f_{,t} + 4p^2 - 2a^2)}{f_{,t} - 2(p^2 - a^2)} . \quad (105)$$

The energy conditions $\mu \pm p \geq 0$ will be satisfied if and only if the right-hand sides of the above equations are positive, i.e.:

$$f_{,t} - 2(p^2 - a^2) > 0 , \quad (106)$$

$$(a^2 - 1)(f_{,t})^2 + (4p^2 - 8a^2)f_{,t} + 4a^2 - 4(p^2 - a^2)^2 > 0 , \quad (107)$$

$$\frac{3}{2}(f_{,t})^2 + (-3p^2 + 2a^2 - 2)f_{,t} + 4p^2 - 2a^2 \geq 0 , \quad (108)$$

and these, in turn, can be seen to imply (after a careful analysis):

$$a^2 - 1 \geq 0 \quad f_{,t} > \beta \quad (109)$$

$$a^2 - 1 < 0 \quad \beta < f_{,t} < 2 \left(1 + \frac{m^2}{4} \frac{1}{1 - |a|} \right) \quad (110)$$

where

$$\beta \equiv \frac{1}{3} \left[a^2 + 3\frac{m^2}{4} + 5 + \sqrt{(a^2 - 1)^2 + 3\frac{m^2}{4}(2 + 2a^2 + 3\frac{m^2}{4})} \right] . \quad (111)$$

From (103)-(105) it is easy to see that a barotropic equation of state of the form $p = p(\mu)$ is not possible.

The analysis of types *III* and *VI* turns out to be very involved, with many sub-cases arising, and so far, we have not been able to find any solution to the EFEs which

is valid (i.e., satisfies energy conditions) all over the space-time manifold. Since we cannot provide any result in the positive we shall not discuss these cases any further here, so as to keep the present study at a reasonable length.

Family B:

For this family, the metric and CKV X take one of the forms given by (19)-(21), and the conformal factor Ψ is then given by (22).

From the field equations for a perfect fluid it is easy to see that the metrics (20) and (21) cannot represent a perfect fluid space-time with $(\mu + p) \neq 0$ and $\mu > 0$; thus, only (19) needs be considered. The field equations (33)-(34) for this metric suggest redefinitions of the coordinates $t = \phi_1(t)$ and $r = \phi_2(r)$, so that the metric takes the form (in the new coordinates):

$$ds^2 = \frac{e^{2f(t-r)}}{\sqrt{H'}} \left\{ -\frac{dt^2}{t} + \frac{H'}{H} dr^2 + 4tdy^2 + Hdz^2 \right\}, \quad (112)$$

where $H = H(r)$, as usual a dash indicates a derivative with respect to r and $f = f(t - r)$ is a function of $t - r \equiv x$. The CKV X and conformal factor Ψ are in these coordinates:

$$X = e^y \left(t^{1/2} \partial_t - \frac{1}{2} t^{-1/2} \partial_y \right), \quad (113)$$

$$\Psi = e^y \sqrt{t} f_{,x}. \quad (114)$$

The remaining field equations are now:

$$(f_{,x}^3 - f_{,x} f_{,xx}) \Sigma_0(r) + f_{,xx} \Sigma_1(r) + f_{,x}^2 \Sigma_2(r) = 0, \quad (115)$$

where

$$\Sigma_0(r) \equiv 2 \frac{H''}{H'}, \quad \Sigma_1(r) \equiv -\frac{5}{4} \left(\frac{H''}{H'} \right)^2 + \frac{H'''}{H'}, \quad (116)$$

$$\Sigma_2(r) \equiv \frac{3}{2} \left(\frac{H''}{H'} \right)^2 - \frac{H'''}{H'} , \quad (117)$$

and they satisfy: $\Sigma_1 + \Sigma_2 = \frac{\Sigma_0^2}{16}$.

Also notice that one of the trivial solutions to (115), namely $f_{,x} = 0$ would correspond to X being a KV, on account of the form of the conformal factor (114) and that the case $f_{,x} = \text{constant}$ leads to an incorrect signature of the metric. Excluding these cases, Σ_0 , Σ_1 and Σ_2 must be constants, and the following possibilities then arise:

(B-1) $\Sigma_0 = 0$, then necessarily $\Sigma_1 = \Sigma_2 = 0$, and

$$H = K^2 r + C , \quad K, C = \text{const} , \quad (118)$$

where we can always make $C = 0$ by suitably redefining r , and f is then a completely arbitrary function of its argument. The density, pressure and velocity field can then be obtained from:

$$\mu \frac{g_{yy}}{4t} = 3\{f_{,x} + (t-r)f_{,x}^2\}, \quad (119)$$

$$(\mu + p) \frac{g_{yy}}{4t} = 2(t-r)\{f_{,x}^2 - f_{,xx}\}, \quad (120)$$

$$(\mu - p) \frac{g_{yy}}{4t} = 2\{(t-r)f_{,xx} + 2(t-r)f_{,x}^2 + 3f_{,x}\}, \quad (121)$$

$$(\mu + p)u_t^2 = 2\{f_{,x}^2 - f_{,xx}\}. \quad (122)$$

From the above expressions one can readily see that no function $f(x)$ exists such that μ , $\mu + p$ and $\mu - p$ are positive over all of the space-time.

(B-2) $\Sigma_0 \neq 0$. This leads to

$$H = K^2 \{e^{4sr} + h_0\} , \quad (123)$$

where K , s and h_0 are arbitrary constants. The function f is then given implicitly

by

$$e^{s/f_{,x}} \left\{ 1 + \frac{s}{f_{,x}} \right\} = e^{-2sx}, \quad (124)$$

and the density and pressure can be obtained from:

$$\mu \frac{g_{yy}}{4t} = \frac{3e^{-4sr}}{4s} \left\{ e^{4sr} \left[f_{,x}^2 (4st - 1) + 2sf_{,x} + \frac{s^2}{3} \right] - h_0(s + f_{,x})^2 \right\}, \quad (125)$$

$$\begin{aligned} (\mu + p) \frac{g_{yy}}{4t} &= \frac{e^{-4sr}}{2(2f_{,x} + s)} \left\{ e^{4sr} \left[-f_{,x}^2 (4st - 1) + 2sf_{,x} + s^2 \right] \right. \\ &\quad \left. + h_0(s + f_{,x})^2 \right\}, \end{aligned} \quad (126)$$

$$\begin{aligned} (\mu - p) \frac{g_{yy}}{4t} &= \frac{e^{-4sr}}{s(2f_{,x} + s)} \left\{ 3e^{4sr} f_{,x} \left[f_{,x}^2 (4st - 1) + \frac{4s}{3}(2st + 1)f_{,x} + s^2 \right] \right. \\ &\quad \left. - h_0 \left[3f_{,x}^3 + 8sf_{,x}^2 + 7s^2 f_{,x} + 2s^3 \right] \right\}. \end{aligned} \quad (127)$$

The energy conditions $\mu > 0$, $\mu \pm p > 0$ will be satisfied if and only if the above expressions are all positive, and a careful analysis of these conditions taking into account (124) shows that they can only hold in some open domains of the space-time manifold (e.g., for $h_0 = 0$ the energy conditions can only hold in the region $t - r < 0$ if $s < 0$, and in the region $-\mathcal{K}^2(t) < t - r < 0$ if $s > 0$, \mathcal{K} being some function of t).

5.2 Spacelike conformal orbits.

Family A:

The forms of the metric functions are then (11), (13), (15) and (16), with $w = 0$ and the coordinates t and r interchanged. We note that the solutions are not simply obtained from (5.1) by interchanging t and r . We next briefly discuss a few solutions belonging to some of the Bianchi types.

Type I: The analysis of this type follows much along the same lines as its coun-

terpart in the case of timelike conformal orbits. Thus we have

$$ds^2 = \frac{e^{f(r)-q(t)}}{\dot{h}(t)} \left\{ -dt^2 + dr^2 \right\} + \frac{e^{f(r)}}{\dot{h}(t)} \left\{ e^{-h(t)} dy^2 + e^{h(t)} dz^2 \right\}, \quad (128)$$

where a dot indicates differentiation with respect to t . Then the independent Einstein equation takes the form:

$$0 = \hat{\Sigma}_0 + \hat{\Sigma}_2(f_{,r})^2 + \hat{\Sigma}_3 f_{,rr}, \quad (129)$$

where the expressions of the $\hat{\Sigma}_i$ s can be formally obtained from those of the Σ_i s given by Eqns. (70), (72) and (73), by simply changing r by t there (i.e., changing primes into dots).

Since $f_{,rr}$ is non-null, the following possibilities arise:

(I.a) $\hat{\Sigma}_3 = 0$, then necessarily $\hat{\Sigma}_0 = \hat{\Sigma}_1 = 0$. In this subcase there are no perfect fluid solutions because the Einstein tensor do not have any timelike eigenvector.

(I.b) $\hat{\Sigma}_3 \neq 0$, then $\hat{\Sigma}_2 = -c\hat{\Sigma}_3$, $\hat{\Sigma}_0 = \alpha\hat{\Sigma}_3$ and

$$f_{,rr} = c(f_{,r})^2 - \alpha, \quad (130)$$

where c and α are arbitrary constants. The first equation yields

$$0 = (1-2c)(\dot{h})^2 + (\dot{q})^2 + (1+2c)\frac{\ddot{h}}{\dot{h}}\dot{q} + 2(1-c)\left(\frac{\ddot{h}}{\dot{h}}\right)^2 - (1-2c)\ddot{q} - 2(1-2c)\frac{h^{..}}{\dot{h}} \quad (131)$$

The case $c = \frac{1}{2}$ can be easily seen to correspond to one of the previously studied cases, namely those with the fluid flow orthogonal to the conformal orbits.

Following now a similar procedure to the one outlined in the case of timelike conformal orbits, the field equations can then be written as

$$\dot{q} = 2\alpha\frac{\sigma}{\dot{\sigma}} + \frac{k}{\dot{\sigma}}, \quad k = \text{const}, \quad (132)$$

$$0 = \dot{\sigma}\ddot{\sigma} + \frac{-1+3c}{1-2c}(\ddot{\sigma})^2 - \frac{2c}{1-2c}\ddot{\sigma}(2\alpha\sigma+k) - \alpha(\dot{\sigma})^2 + \frac{1}{2(1-2c)}(2\alpha\sigma+k)^2 + \frac{1}{2}, \quad (133)$$

with $\sigma(t)$ defined through

$$\dot{\sigma} = \frac{1}{h}. \quad (134)$$

Explicit expressions for μ , p and u_t can be readily derived from:

$$(\mu + p)u_t^2 = \frac{1}{2(1-2c)} \left[2\alpha \frac{\sigma}{\dot{\sigma}} + \frac{k}{\dot{\sigma}} - \frac{\ddot{\sigma}}{\dot{\sigma}} \right]^2, \quad (135)$$

$$(\mu - p)g_{rr} = -(1+c)(f_{,r})^2 + \alpha + \frac{\ddot{\sigma}}{\dot{\sigma}}, \quad (136)$$

$$(\mu + p)g_{rr} = \frac{1}{2(1-2c)} \left[2\alpha \frac{\sigma}{\dot{\sigma}} + \frac{k}{\dot{\sigma}} - \frac{\ddot{\sigma}}{\dot{\sigma}} \right]^2 - \frac{1-2c}{2}(f_{,r})^2. \quad (137)$$

Now, from all the possible solutions for $f(r)$ to equation (130), the energy conditions $\mu \pm p > 0$ single out

$$\alpha = c\beta^2, \quad f = -\frac{1}{c} \ln(\cosh c\beta r) \quad (138)$$

and restrict c to values $c < 1/2$.

A particular solution to (133) which satisfies the energy conditions is:

$$\sigma = \frac{\cosh \beta t}{\beta^2}, \quad k = -(1-2c), \quad (139)$$

and we notice that the solution is valid for $c > -1$ and in order for the metric to have the correct signature, t must be positive.

Type V: There are no perfect fluid solutions of this type.

Family B:

An inspection of the field equations suggests, as in the case of timelike conformal orbits, a redefinition of t and r , so that in the new coordinates the metric, CKV X

and the conformal factor Ψ take the following forms:

$$ds^2 = \frac{2e^{2f(x)}e^{-kt/2}}{\sqrt{\dot{H}}} \left\{ -\frac{1}{4} \frac{\dot{H}}{H} dt^2 + \frac{k}{4} \frac{ce^{kr}}{1+ce^{kr}} dr^2 + \frac{1+ce^{kr}}{k} dy^2 + H dz^2 \right\}, \quad (140)$$

$$X = e^y \left\{ e^{-kr/2} \sqrt{1+ce^{kr}} \partial_r - \frac{k}{2} \frac{ce^{kr/2}}{\sqrt{1+ce^{kr}}} \partial_y \right\}, \quad (141)$$

$$\Psi = -e^y f_{,x} e^{-kr/2} \sqrt{1+ce^{kr}}, \quad (142)$$

where k and c are constants and x is defined as $x \equiv t - r$. The field equations then reduce to:

$$2(f_{,x}f_{,xx} - f_{,x}^3)\Sigma_0 + f_{,x}^2\Sigma_1 + (f_{,xx} + \frac{k}{2}f_{,x})\Sigma_2 = 0, \quad (143)$$

where $\Sigma_0(t)$, $\Sigma_1(t)$ and $\Sigma_2(t)$ are given by:

$$\Sigma_0 \equiv -k + \frac{\ddot{H}}{\dot{H}}, \quad (144)$$

$$\Sigma_1 \equiv -k^2 + \frac{k}{2} \frac{\ddot{H}}{\dot{H}} + \frac{1}{2} \left(\frac{\ddot{H}}{\dot{H}} \right)^2 - \left(\frac{\ddot{H}}{\dot{H}} \right) \cdot = -\dot{\Sigma}_0 + \frac{1}{2} \Sigma_0^2 + \frac{3}{2} k \Sigma_0, \quad (145)$$

$$\Sigma_2 \equiv \frac{k^2}{4} - \frac{1}{4} \left(\frac{\ddot{H}}{\dot{H}} \right)^2 + \left(\frac{\ddot{H}}{\dot{H}} \right) \cdot = \dot{\Sigma}_0 - \frac{1}{4} \Sigma_0 (\Sigma_0 + 2k). \quad (146)$$

The possibility $f_{,x} = \text{const} \neq 0$ leads to the wrong Segre type of the Einstein tensor (i.e., no perfect fluid solutions exist). Then, from Eqn.(143) it is immediate that either $f_{,x} = 0$ (and then X becomes a KV; see (142)) or Σ_0 and Σ_1 are both constants, which, in turn, implies:

$$H = M e^{dt} + m, \quad (147)$$

where M , m and d are constants. Substituting this back into the field equation (143) we obtain:

$$-2(d^2 - k^2)f_{,xx} + 16(d-k)(f_{,x}f_{,xx} - f_{,x}^3) + 4(d^2 - k^2 + k(d-k))f_{,x}^2 - k(d^2 - k^2)f_{,x} = 0, \quad (148)$$

and we can distinguish the following cases:

(i) $f_{,xx} = 0$; i.e., $f_{,x}$ =constant, and from Eqn.(148) we obtain

$$f_{,x} = \frac{k+d}{4} \quad \text{or} \quad f_{,x} = \frac{k}{4}. \quad (149)$$

In the first case ($f_{,x} = \frac{k+d}{4}$) the Segre type does not correspond to a perfect fluid (since $u_t^2 = 0$) and in the second case $(\mu + p)u_t^2 < 0$, so this cannot represent a physical perfect fluid.

(ii) $f_{,xx} \neq 0$; whence Eqn.(148) can be integrated once, leading to an implicit expression for $f_{,x}$. A careful analysis of the energy conditions $\mu \pm p \geq 0$ shows that in the general case (i.e., no assumptions on the values of the parameters k and d), they *can only* be satisfied over certain restricted open domains of the space-time. However, two special cases arise which deserve special attention; namely, $k = d$ and $k = -d$. For $k = d$, Eqn.(148) is identically satisfied, f thus being a completely arbitrary function; however, this does not correspond to a perfect fluid (it is of the wrong Segre type). In the second case, Eqn.(148) can be integrated to give:

$$f = \ln \frac{e^{kx/4}}{\frac{4}{k} + ae^{kx/4}}, \quad (150)$$

a being a constant, and μ , p and u_t can then be obtained from:

$$(\mu + p) = \frac{e^{-kx/2}}{c\sqrt{-kM}} (4 + ak e^{kx/4})(-cm e^{kt} - Me^{-kr} - 2cM), \quad (151)$$

$$\begin{aligned} (\mu - p) = & \frac{e^{-kx/2}}{c\sqrt{-kM}} [(16 + ak e^{kx/4})(-cm e^{kt} - Me^{-kr}) \\ & + ackM e^{kx/4}(2 - ak e^{kx/4})], \end{aligned} \quad (152)$$

$$(\mu + p)u_t^2 = \frac{-k^2}{2(4 + ak e^{kx/4})}. \quad (153)$$

As usual, the energy conditions will be fulfilled if and only if the above expressions are all positive and, again, this can only be possible over certain open domains of the

manifold (notice that k must be positive in order for the metric to have the correct signature).

6 Discussion

We have studied perfect fluid space-times admitting a three-dimensional Lie group of conformal motions containing a two-dimensional Abelian subgroup of isometries. All such space-times have been classified geometrically and in each class the metric has been explicitly given in coordinates adapted to the symmetry vectors.

In section 3 we restricted attention to the diagonal case, and in section 4 we found all such perfect fluid solutions in which the fluid four-velocity is tangential or orthogonal to the conformal orbits. In the former case the orbits are necessarily timelike and the only solutions for which \mathcal{C}_3 is maximal and X is proper are of type *III* in family A and given by Eqns. (36)-(42). In the latter case, in which u^a is orthogonal to spacelike conformal orbits, there is a 2-parameter family of solutions given by (44)-(46) of type *I* which are valid for $t > 0$ and are separable in t and r (see [12]); and a class of solutions of type *VI* [see Eqns. (47)-(52) and Eqns. (53)-(55) or (56)-(58)]. In this latter case, physical constraints (e.g., the energy conditions) restrict the validity of the solutions to a region of the space-time and, again, there are no solutions in family B.

In section 5 perfect fluid solutions were sought in the general (tilting) case in which the fluid four-velocity is neither tangential to nor orthogonal to the conformal orbits. We chose to work in coordinates adapted to the CKV and again the two cases in which the conformal orbits are timelike (subsection 5.1) or spacelike (subsection 5.2) were considered.

In the timelike case, solutions in family A of types *I* [metric (67), field eqns.

(91)-(96)] and V [metric (99), field eqns. (103)-(105)] were obtained. In both cases solutions exist such that the energy conditions are satisfied on the whole space-time manifold; a particular solution of type I was given by Eqns. (97) and (98). We noted that solutions of type V cannot admit an equation of state of the form $p = p(\mu)$. In the spacelike case, solutions in family A of type I were found and a particular solution, given by Eqn. (139), in which the energy conditions are always satisfied, was displayed. There are no solutions of type V in this case.

All solutions in family B can be found in both the timelike case [see Eqns. (112)-(124)] and the spacelike case [see Eqns. (140)-(150)]. However, our analysis showed that in general there are no solutions for which the energy conditions are satisfied over the entire space-time manifold.

The case of null conformal orbits will be studied in a future paper.

Acknowledgments

The authors would like to thank M. Mars (Univ. of Barcelona) for many helpful discussions and suggestions. They would also like to thank the referees for many helpful suggestions which have greatly contributed to making the paper more precise and readable. One of the authors (AAC) would like to thank the Spanish Government and the “Dirección General de Investigación Científica y Técnica (DGICYT)” for financial assistance and the Department of Physics at the Universitat de les Illes Balears for its warm hospitality. This work is partially supported by the DGICYT Project No. PB 91-0335. Financial support from STRIDE program (Research Project No. STRDB/C/CEN/509/92) is also acknowledged.

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